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DAVENPORT'S METHOD AND SLIM EXCEPTIONAL SETS: THE ASYMPTOTIC FORMULAE IN WARING'S PROBLEM

KOICHI KAWADA AND TREVOR D. WOOLEY*

Abstract. We apply a method of Davenport to improve several estimates for slim exceptional sets associated with the asymptotic formula in Waring's problem. In particular, we show that the anticipated asymptotic formula in Waring's problem for sums of seven cubes holds for all but $O(N^{1/3+\varepsilon})$ of the natural numbers not exceeding N.

1. Introduction

Earlier work concerning slim exceptional sets in Waring's problem is based on the introduction of an exponential sum over the exceptional set, and a subsequent analysis of auxiliary mean values involving the latter generating function (see [17], [18], [19], [9], [20]). Such a strategy replaces the application of Bessel's inequality conventionally applied within the Hardy-Littlewood (circle) method. Loosely speaking, the newer methods show that when exceptional sets are small, then they are necessarily *very* small, and an obstruction to further progress is the difficulty of establishing the former prerequisite. An old method of Davenport [5] is based on a Diophantine interpretation of the application of Cauchy's inequality restricted to thin sequences. Our goal in this paper is to show how Davenport's method may be applied to good effect in deriving slim exceptional set estimates, thereby expanding the catalogue of problems accessible to slim technology.

When s and k are natural numbers, we denote by $R_{s,k}(n)$ the number of representations of a positive integer n as the sum of s kth powers of positive integers. A heuristic application of the circle method suggests that for $k \geqslant 3$ and $s \geqslant k+1$, one should have the asymptotic relation

$$R_{s,k}(n) = \frac{\Gamma(1+1/k)^s}{\Gamma(s/k)} \mathfrak{S}_{s,k}(n) n^{s/k-1} + o(n^{s/k-1}), \tag{1.1}$$

where

$$\mathfrak{S}_{s,k}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \Big(q^{-1} \sum_{r=1}^{q} e(ar^k/q) \Big)^s e(-an/q),$$

and $e(z)$ denotes $\exp(2\pi i z)$. It is worth noting here that, under modest congruence conditions, one has $1 \ll \mathfrak{S}_{s,k}(n) \ll n^{\varepsilon}$, and thus the conjectural relation

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(1.1) may be interpreted as an honest asymptotic formula (see sections 4.3, 4.5 and 4.6 of [14] for details). We measure the frequency with which the formula (1.1) fails by defining an associated exceptional set as follows. When $\psi(t)$ is a function of a positive variable t, we denote by $\widetilde{E}_{s,k}(N;\psi)$ the number of integers n, with $1\leqslant n\leqslant N$, for which

$$\left| R_{s,k}(n)-\frac{\Gamma(1+1/k)^s}{\Gamma(s/k)}\mathfrak{S}_{s,k}(n)n^{s/k-1}\right| > n^{s/k-1}\psi(n)^{-1}. \tag{1.2}$$

By applying classical methods based on the use of Bessel's inequality, one may derive from work of Vaughan [12] and [13] the estimate

$$\widetilde{E}_{s,k}(N;\psi)\ll N^{1-(s2^{2-k}-2)/k}(\log N)^{-\nu}\psi(N)^2\quad (2^{k-1}\leqslant s\leqslant 2^k), \tag{1.3}$$

valid whenever $\psi(N)$ grows sufficiently slowly, with the exponent $\nu=\nu(s,k)$ positive when $s=2^{k-1}$. This work also establishes that when $\psi(t)$ grows no faster than a suitable power of $\log t$, then $\widetilde{E}_{s,k}(N;\psi)\ll 1$ for $s\geqslant 2^k$. In §2 we improve on the upper bound (1.3) whenever $s>\frac{3}{4}2^k$. For ease of future reference, we summarise our new conclusions followed by those previously available separately for each exponent k. It is convenient here, and in what follows, to refer to a function $\psi(t)$ as being a *sedately increasing function* when $\psi(t)$ is a function of a positive variable t, increasing monotonically to infinity, and satisfying the condition that when t is large, one has $\psi(t)=O(t^\delta)$ for a positive number δ sufficiently small in the ambient context.

Theorem 1.1. *Suppose that $\psi_3(t)$ is a sedately increasing function. Then for each $\varepsilon>0$, one has $\widetilde{E}_{7,3}(N;\psi_3)\ll N^{1/3+\varepsilon}\psi_3(N)^4$.*

For comparison, the relation (1.3), which in this case yields a bound of quality $\widetilde{E}_{s,3}(N;\psi)\ll N^{1-(s-4)/6}\psi_3(N)^2$ $(4\leqslant s\leqslant 7)$, supplies the estimate $\widetilde{E}_{7,3}(N;\psi_3)\ll N^{1/2}\psi_3(N)^2$. This was improved in Theorem 1.3 of [18], so that whenever $\psi(t)=O((\log t)^{1-\delta})$ for some $\delta>0$, then $\widetilde{E}_{7,3}(N;\psi)\ll N^{4/9+\varepsilon}$. The conclusion of Theorem 1.1 is superior to both estimates.

Theorem 1.2. *Suppose that $\psi_4(t)$ is a sedately increasing function. Then, for each $\varepsilon>0$, one has $\widetilde{E}_{s,4}(N;\psi_4)\ll N^{\alpha_s+\varepsilon}\psi_4(N)^4$ $(s=13,14)$, where $\alpha_{13}=\frac{5}{8}$ and $\alpha_{14}=\frac{1}{2}$.*

The conclusion of Theorem 1.1 of [20] provides a bound of the same shape as that supplied by this theorem when $s=15$, though with $\alpha_{15}=\frac{7}{16}$ and the factor $\psi_4(N)^4$ replaced by $\psi_4(N)^2$. Meanwhile, the earlier bound (1.3) in this case yields an estimate of the latter type with $\alpha_s=1-\frac{s-8}{16}$ $(8\leqslant s\leqslant 16)$.

Theorem 1.3. *Suppose that $\psi_5(t)$ is a sedately increasing function. Then, for each $\varepsilon>0$, one has $\widetilde{E}_{s,5}(N;\psi_5)\ll N^{\beta_s+\varepsilon}\psi_5(N)^4$ $(25\leqslant s\leqslant 28)$, where $\beta_s=\frac{4}{5}-\frac{s-24}{20}$.*

A bound of the same type is supplied by Theorem 1.2 of [20], though with $\beta_{29}=\frac{23}{40}$, $\beta_{30}=\frac{11}{20}$ and $\beta_{31}=\frac{3}{8}$, and the factor $\psi_5(N)^4$ replaced by $\psi_5(N)^2$.

Meanwhile, the estimate (1.3) yields analogous bounds with $\beta_s = 1 - \frac{s-16}{40}$ $(16 \leqslant s \leqslant 32)$.

Theorem 1.4. *Suppose that $\psi_6(t)$ is a sedately increasing function. Then, for each $\varepsilon > 0$, one has $\widetilde{E}_{s,6}(N;\psi_6) \ll N^{\gamma_s+\varepsilon}\psi_6(N)^4$ $(44 \leqslant s \leqslant 51)$, where $\gamma_s = \frac{5}{6} - \frac{s-43}{48}$.*

An estimate of this shape is delivered by Theorem 1.3 of [20], though with $\gamma_s = \frac{2}{3} - \frac{s-51}{96}$ $(52 \leqslant s \leqslant 55)$, and the factor $\psi_6(N)^4$ replaced by $\psi_6(N)^2$. Meanwhile, a careful application of the methods of Heath-Brown [7] and Boklan [1] yields bounds of this type with $\gamma_s = 1 - \frac{s-28}{72}$ $(28 \leqslant s \leqslant 31)$ and $\gamma_s = 1 - \frac{s-27}{96}$ $(32 \leqslant s \leqslant 55)$. The main conclusion of [1], in particular, shows that $\widetilde{E}_{s,6}(N;\psi_6) \ll 1$ when $s \geqslant 56$, provided that $\psi_6(t) = O((\log t)^\delta)$ for a suitably small positive number δ.

Theorem 1.5. *Suppose that $\psi_7(t)$ is a sedately increasing function. Then, for each $\varepsilon > 0$, one has $\widetilde{E}_{s,7}(N;\psi_7) \ll N^{\delta_s+\varepsilon}\psi_7(N)^4$ $(85 \leqslant s \leqslant 100)$, where $\delta_s = \frac{6}{7} - \frac{s-84}{112}$.*

For comparison, Theoren 1.4 of [20] delivers an estimate of this shape with $\delta_s = \frac{5}{7} - \frac{s-100}{224}$ $(101 \leqslant s \leqslant 108)$ and $\delta_s = \frac{4}{7} - \frac{s-108}{224}$ $(109 \leqslant s \leqslant 111)$, though with the factor $\psi_7(N)^4$ again replaced by $\psi_7(N)^2$. Meanwhile, the methods of [7] and [1] yield bounds of this type with $\delta_s = 1 - \frac{s-56}{168}$ $(56 \leqslant s \leqslant 68)$ and $\delta_s = 1 - \frac{s-52}{224}$ $(69 \leqslant s \leqslant 111)$. Also, one finds from [1] that $\widetilde{E}_{s,7}(N;\psi_7) \ll 1$ for $s \geqslant 112$, provided that $\psi_7(t) = O((\log t)^\delta)$ and $\delta > 0$ is suitably small.

Theorem 1.6. *Suppose that $\psi_8(t)$ is a sedately increasing function. Then, for each $\varepsilon > 0$, one has $\widetilde{E}_{s,8}(N;\psi_8) \ll N^{\eta_s+\varepsilon}\psi_8(N)^4$, where $\eta_s = \frac{7}{8} - \frac{s-168}{192}$ $(171 \leqslant s \leqslant 180)$, and $\eta_s = \frac{7}{8} - \frac{s-164}{256}$ $(181 \leqslant s \leqslant 196)$.*

The conclusion of Theorem 1.5 of [20] in this instance delivers an estimate of the above shape with $\eta_s = \frac{3}{4} - \frac{s-196}{512}$ $(197 \leqslant s \leqslant 212)$, $\eta_s = \frac{5}{8} - \frac{s-212}{512}$ $(213 \leqslant s \leqslant 220)$, and $\eta_s = \frac{1}{2} - \frac{s-220}{512}$ $(221 \leqslant s \leqslant 223)$, though again with the factor $\psi_8(N)^4$ replaced by $\psi_8(N)^2$. Meanwhile, the methods of [7] and [1] may be wrought to provide estimates of the latter type with $\eta_s = 1 - \frac{s-112}{384}$ $(112 \leqslant s \leqslant 148)$ and $\eta_s = 1 - \frac{s-100}{512}$ $(149 \leqslant s \leqslant 223)$. In addition, the bound $\widetilde{E}_{s,8}(N;\psi_8) \ll 1$, for $s \geqslant 224$, follows from [1] provided that $\psi_8(t) = O((\log t)^\delta)$ for a suitably small positive number δ.

When $k \geqslant 9$, our methods are again of use in estimating $\widetilde{E}_{s,k}(N;\psi_k)$ when s is relatively large, though Vinogradov's methods increasingly dominate the analysis and transform the landscape (see [6] and [15] for the relevant ideas). We therefore avoid discussion of the situation for larger values of k.

As our next application of Davenport's method interpreted through slim technology, we consider higher moments of the counting functions $R_{s,k}(n)$. In order to illustrate ideas, we concentrate on sums of cubes, and in §3 derive an improvement on recent work of Brüdern and the second author [4].

Theorem 1.7. *For any positive number h smaller than $\frac{7}{2}$, there is a positive number $\delta = \delta(h)$ with the property that*

$$\sum_{1\leqslant n\leqslant N} \Big| R_{5,3}(n) - \frac{\Gamma(\frac{4}{3})^5}{\Gamma(\frac{5}{3})}\mathfrak{S}_{5,3}(n)n^{2/3}\Big|^h \ll N^{2h/3+1-\delta}.$$

In addition, for each $\varepsilon > 0$, one has

$$\sum_{1\leqslant n\leqslant N} \Big| R_{5,3}(n) - \frac{\Gamma(\frac{4}{3})^5}{\Gamma(\frac{5}{3})}\mathfrak{S}_{5,3}(n)n^{2/3}\Big|^3 \ll N^{35/12+\varepsilon}. \tag{1.4}$$

The first conclusion of this theorem includes Theorem 1.1 of [4] as the special case in which $h = 3$. The second estimate, meanwhile, has the same strength as Theorem 1.2 of [4], in which it is asserted that

$$\sum_{1\leqslant n\leqslant N} \Big(R_{5,3}(n) - \frac{\Gamma(\frac{4}{3})^5}{\Gamma(\frac{5}{3})}\mathfrak{S}_{5,3}(n)n^{2/3}\Big)^3 \ll N^{35/12+\varepsilon}.$$

Our conclusion here has greater content, and also supersedes the conclusion of Theorem 1.1 of [4], in which the bound (1.4) is obtained with the right hand side replaced by $N^3(\log N)^{\varepsilon-4}$.

Before announcing our final application, we require some additional notation. When P and R are real numbers with $1 \leqslant R \leqslant P$, we define the set of smooth numbers $\mathcal{A}(P,R)$ by

$$\mathcal{A}(P,R) = \{n \in [1,P]\cap\mathbb{Z} : p \text{ prime and } p|n \Rightarrow p \leqslant R\}.$$

We then define the exponential sum $h(\alpha) = h(\alpha; P,R)$ by

$$h(\alpha;P,R) = \sum_{y\in\mathcal{A}(P,R)} e(\alpha y^3).$$

The sixth moment of the latter sum has played an important role in a plethora of recent applications. Write $\tau = (213 - 4\sqrt{2833})/164 = 1/1703.6\ldots$. Then as a consequence of the work of the second author [16], given any $\varepsilon > 0$, there exists a positive number $\eta = \eta(\varepsilon)$ with the property that whenever $1 \leqslant R \leqslant P^\eta$, one has

$$\int_0^1 |h(\alpha;P,R)|^6\, d\alpha \ll P^{13/4-\tau+\varepsilon}. \tag{1.5}$$

We henceforth assume that whenever R appears in a statement, either implicitly or explicitly, then $1 \leqslant R \leqslant P^\eta$ with η a positive number sufficiently small in the context of the upper bound (1.5). Finally, when $\mathcal{B} \subseteq [P/2,P]\cap\mathbb{Z}$, we define the exponential sum $F(\alpha) = F(\alpha;\mathcal{B})$ by

$$F(\alpha;\mathcal{B}) = \sum_{x\in\mathcal{B}} e(\alpha x^3).$$

Theorem 1.8. *Suppose that c_i, d_i $(i = 1,2,3)$ are integers satisfying the condition*

$$(c_1d_2 - c_2d_1)(c_1d_3 - c_3d_1)(c_2d_3 - c_3d_2) \neq 0.$$

Write $\lambda_j = c_j\alpha + d_j\beta$ $(j=1,2,3)$. *Also, let* $\mathcal{B} \subseteq [P/2, P] \cap \mathbb{Z}$. *Then for each* $\varepsilon > 0$, *there exists a positive number* $\eta = \eta(\varepsilon)$ *such that, whenever* $1 \leqslant R \leqslant P^\eta$, *one has the estimate*

$$\int_0^1 \int_0^1 \prod_{i=1}^3 |F(\lambda_i)^2 h(\lambda_i)^2|\, d\alpha\, d\beta \ll P^{49/8-3\tau/2+\varepsilon}.$$

In addition, one has

$$\int_0^1 \int_0^1 |h(\lambda_1)h(\lambda_2)h(\lambda_3)|^4\, d\alpha\, d\beta \ll P^{49/8-3\tau/2+\varepsilon}.$$

For comparison, Theorem 4 of [3] contains a similar conclusion to the second estimate of Theorem 1.8, save that our exponent $\frac{49}{8} - \frac{3}{2}\tau$ is there replaced by $\frac{25}{4} - \tau$. The twelfth moment estimate supplied by Theorem 4 of [3] was employed, together with its close kin, so as to establish the validity of the Hasse principle for pairs of diagonal cubic equations in thirteen or more variables. We are not aware of additional applications stemming from Theorem 1.8, though quantitative improvements in potential effective bounds for solutions ought to benefit from our sharper estimate.

Throughout, the letter ε will denote a sufficiently small positive number. We use $\ll$ and $\gg$ to denote Vinogradov's well-known notation, implicit constants depending at most on ε, unless otherwise indicated. In an effort to simplifiy our analysis, we adopt the convention that whenever ε appears in a statement, then we are implicitly asserting that for each $\varepsilon > 0$, the statement holds for sufficiently large values of the main parameter. Note that the "value" of ε may consequently change from statement to statement, and hence also the dependence of implicit constants on ε. Finally, from time to time we make use of vector notation in order to save space. Thus, for example, we may abbreviate $(c_1, \ldots, c_t)$ to $\mathbf{c}$.

2. The asymptotic formula in Waring's problem

Our initial approach to the task of proving the first six theorems follows closely that taken in the second author's previous work [20]. Initially, we consider integers k and s with $3 \leqslant k \leqslant 8$ and $s \geqslant \frac{3}{4}2^k$. Suppose that N is a large positive number, and let $\psi = \psi_k(t)$ be a sedately increasing function. We denote by $\mathcal{Z}_{s,k}(N)$ the set of integers n with $N/2 < n \leqslant N$ for which the inequality (1.2) holds, and we abbreviate $\operatorname{card}(\mathcal{Z}_{s,k}(N))$ to $Z = Z_{s,k}$.

Write $P = P_k$ for $[N^{1/k}]$, and define the exponential sum $f(\alpha) = f_k(\alpha)$ by

$$f_k(\alpha) = \sum_{1 \leqslant x \leqslant P_k} e(\alpha x^k).$$

Also, let $\mathfrak{M} = \mathfrak{M}_k$ denote the union of the intervals

$$\mathfrak{M}_k(q,a) = \{\alpha \in [0,1) : |q\alpha - a| \leqslant (2k)^{-1} P_k N^{-1}\},$$

with $0 \leqslant a \leqslant q \leqslant (2k)^{-1} P_k$ and $(a,q) = 1$, and define $\mathfrak{m} = \mathfrak{m}_k$ by putting $\mathfrak{m}_k = [0,1) \setminus \mathfrak{M}_k$. Then the argument of [20] leading to equation (2.5) reveals

that there exist complex numbers $\eta_n = \eta_n(s,k)$ with $|\eta_n| = 1$, satisfying the condition that, with the exponential sum $K(\alpha) = K_{s,k}(\alpha)$ defined by

$$K_{s,k}(\alpha) = \sum_{N/2<n\leqslant N} \eta_n(s,k)e(n\alpha),$$

one has

$$\int_{\mathfrak{m}} |f_k(\alpha)^s K_{s,k}(\alpha)|\,d\alpha \gg N^{s/k-1}\psi_k(N)^{-1}Z_{s,k}. \tag{2.1}$$

Our goal is now to obtain an upper bound for the integral on the left hand side of the relation (2.1), and thereby obtain an upper bound on $Z_{s,k}$. This we achieve by exploiting an estimate whose roots go back to a method of Davenport [5].

Lemma 2.1. *Let k be a natural number with $k \geqslant 3$, and suppose that $1 \leqslant j \leqslant k-2$. Then one has*

$$\int_0^1 |f_k(\alpha)^{2^j} K_{s,k}(\alpha)^2|\,d\alpha \ll P_k^{2^j-1}Z_{s,k} + P_k^{2^j-j/2-1+\varepsilon}Z_{s,k}^{3/2}.$$

Proof. The claimed estimate is immediate from the conclusion of Lemma 6.1 of [10]. □

Suppose now that l is a natural number, and put $s = \frac{3}{4}2^k + l$. Then an application of Schwarz's inequality shows that

$$\begin{aligned}\int_{\mathfrak{m}} |f(\alpha)^s K(\alpha)|\,d\alpha \leqslant \Big(\sup_{\alpha\in\mathfrak{m}} |f(\alpha)|\Big)^{2^{k-3}+l} \Big(\int_0^1 |f(\alpha)|^{2^k}\,d\alpha\Big)^{1/2}\\ \times \Big(\int_0^1 |f(\alpha)^{2^{k-2}}K(\alpha)^2|\,d\alpha\Big)^{1/2}.\end{aligned} \tag{2.2}$$

But Weyl's inequality (see, for example, Lemma 2.4 of [14]) yields the upper bound

$$\sup_{\alpha\in\mathfrak{m}} |f(\alpha)| \ll P^{1-2^{1-k}+\varepsilon},$$

and Hua's lemma (see Lemma 2.5 of [14]) supplies the estimate

$$\int_0^1 |f(\alpha)|^{2^k}\,d\alpha \ll P^{2^k-k+\varepsilon}.$$

Consequently, applying these estimates in combination with the case $j = k-2$ of Lemma 2.1, we deduce from (2.2) that

$$\begin{aligned}\int_{\mathfrak{m}} |f(\alpha)^s K(\alpha)|\,d\alpha &\ll P^\varepsilon (P^{1-2^{1-k}})^{2^{k-3}+l}(P^{2^k-k})^{1/2}\\ &\quad\times (P^{2^{k-2}-1}Z + P^{2^{k-2}-k/2}Z^{3/2})^{1/2}\\ &\ll P^{s-k-l2^{1-k}+\varepsilon}(P^{k-3/2}Z + P^{(k-1)/2}Z^{3/2})^{1/2}.\end{aligned}$$

Substituting this bound into (2.1), we find that

$$Z \ll \psi_k(N)P^{\varepsilon-l2^{1-k}}(P^{k-3/2}Z + P^{(k-1)/2}Z^{3/2})^{1/2},$$

whence

$$Z \ll P^{k-3/2-l2^{2-k}+\varepsilon}\psi_k(N)^2 + P^{k-1-l2^{3-k}+\varepsilon}\psi_k(N)^4.$$

Since $P \asymp N^{1/k}$, we conclude that

$$Z \ll N^{1-(3+l2^{3-k})/(2k)+\varepsilon}\psi_k(N)^2 + N^{1-(1+l2^{3-k})/k+\varepsilon}\psi_k(N)^4.$$

In particular, when $1 \leqslant l \leqslant 2^{k-3}$, one obtains

$$Z \ll N^{1-(1+l2^{3-k})/k+\varepsilon}\psi_k(N)^4. \tag{2.3}$$

Recall that $s = \frac{3}{4}2^k + l$. Then on summing over dyadic intervals, one finds that the bound (2.3) leads to the estimate

$$\widetilde{E}_{s,k}(N;\psi) \ll N^{1-(s2^{3-k}-5)/k+\varepsilon}\psi_k(N)^4 = N^{\omega_{s,k}+\varepsilon}\psi(N)^4,$$

in which

$$\omega_{s,k} = 1 - \frac{1}{k} - \frac{s-\frac{3}{4}2^k}{k2^{k-3}}.$$

This confirms the estimates claimed in Theorems 1.1, 1.2 and 1.3.

When $k \geqslant 6$, improvements may be wrought via the technology introduced by Heath-Brown [7], and refined by Boklan [1]. The key elements of such an approach, so far as the application at hand is concerned, are contained in the following lemma. In this context, when r is a non-negative integer, we write

$$\Theta_{r,k} = P^{k-2r}\int_0^1 |f(\alpha)^{2r}K(\alpha)^2|\,d\alpha.$$

Lemma 2.2. *Suppose that $k \geqslant 6$, and that s,t,u,v are non-negative integers with*

$$s = \tfrac{7}{16}2^k + t + u \quad \text{and} \quad s = \tfrac{3}{8}2^k + [(k+1)/2] + u + v.$$

Then for each $\varepsilon > 0$, one has

$$\int_{\mathfrak{m}} |f_k(\alpha)^s K_{s,k}(\alpha)|\,d\alpha \ll P_k^{s-k+\varepsilon}(P_k^{-2^{3-k}t/3} + P_k^{-2^{1-k}v})\Theta_{u,k}^{1/2}.$$

Proof. The conclusion of the lemma is an immediate consequence of Lemma 4.1 of [20]. $\square$

It transpires that the methods of this paper do not supersede the classical bounds reported in the introduction when $s \leqslant s_0$, where

$$s_0 = \tfrac{5}{8}2^k + [(k+1)/2].$$

We therefore restrict attention henceforth to the situation with $s > s_0$. We apply Lemma 2.2 with $u = 2^{k-3}$, $t = 2^{k-4} + [(k+1)/2] + l$ for some $l \geqslant 1$, and $v = 2^{k-4} + t - [(k+1)/2]$. Observe that one then has

$$s = s_0 + l = \tfrac{9}{16}2^k + t = 2^{k-1} + [(k+1)/2] + v.$$

Define $l_0 = l_0(k)$ by

$$l_0(k) = \begin{cases} 1, & \text{when } k = 6, 7, \\ 17, & \text{when } k = 8. \end{cases}$$

Then a modest computation reveals that when $l \geqslant l_0$, one has

$$\tfrac{2}{3}t \geqslant \tfrac{2}{3}(2^{k-4} + [(k+1)/2] + l_0) > 2^{k-3} - 2[(k+1)/2],$$

whence

$$\tfrac{8}{3}2^{-k}t > 2^{1-k}(2^{k-4} + t - [(k+1)/2]).$$

It follows that when $l \geqslant l_0$, one has $\frac{8}{3}2^{-k}t > 2^{1-k}v$, and that the reverse inequality holds only when $1 \leqslant l < l_0$. In particular, we deduce from Lemma 2.2 that

$$\int_{\mathfrak{m}} |f(\alpha)^s K(\alpha)|\, d\alpha \ll P^{s-k+\varepsilon}(P^{-w_s})\Theta_{u,k}^{1/2},$$

where $w_s = \frac{8}{3}2^{-k}t$ when $1 \leqslant l < l_0$, and $w_s = 2^{1-k}v$ when $l \geqslant l_0$.

We first examine the situation in which $l \geqslant l_0$, so that $w_s = 2^{1-k}v$. Here, employing Lemma 2.1 with $j = k-2$ in order to estimate $\Theta_{u,k}$, we infer from (2.1) that

$$N^{s/k-1}\psi_k(N)^{-1}Z \ll P^{s-k+\varepsilon}(P^{-w_s})(P^{k-1}Z + P^{k/2}Z^{3/2})^{1/2}.$$

Thus, on recalling that $P = [N^{1/k}]$ and $v = s - 2^{k-1} - [(k+1)/2]$, we deduce that

$$\begin{aligned} Z &\ll P^{k-1-2^{2-k}v+\varepsilon}\psi_k(N)^2 + P^{k-2^{3-k}v+\varepsilon}\psi_k(N)^4 \\ &\ll N^{a(s,k)+\varepsilon}\psi_k(N)^2 + N^{b(s,k)+\varepsilon}\psi_k(N)^4, \end{aligned} \tag{2.4}$$

where

$$a(s,k) = 1 - \frac{1}{k} - \frac{s - 2^{k-1} - [(k+1)/2]}{k2^{k-2}}$$

and

$$b(s,k) = 1 - \frac{1}{k} - \frac{s - 5\cdot 2^{k-3} - [(k+1)/2]}{k2^{k-3}}.$$

Write

$$s_1 = \tfrac{3}{4}2^k + [(k+1)/2].$$

Then one may verify that $b(s,k) \geqslant a(s,k)$ when $s \leqslant s_1$, and otherwise $b(s,k) < a(s,k)$. Hence, by summing over dyadic intervals, we conclude that

$$\widetilde{E}_{s,k}(N) \ll \begin{cases} N^{\omega_{s,k}+\varepsilon}\psi_k(N)^4, & \text{when } s_0 < s \leqslant s_1, \\ N^{\omega_{s,k}+\varepsilon}\psi_k(N)^2, & \text{when } s > s_1, \end{cases}$$

where

$$\omega_{s,k} = 1 - \frac{1}{k} - \frac{s - s_0}{k2^{k-3}},$$

when $s_0 < s \leqslant s_1$, and

$$\omega_{s,k} = 1 - \frac{2}{k} - \frac{s - s_1}{k2^{k-2}},$$

when $s > s_1$. This confirms the upper bounds asserted in Theorems 1.4 and 1.5, and also that asserted in Theorem 1.6 when $s \geqslant 181$.

We turn now to the complementary situation in which $k = 8$ and $1 \leqslant l < l_0$. In this case we have $w_s = \frac{1}{96}t$. Noting the adjustment in the value of w_s, and

recalling that $t = s - 144$, we may proceed exactly as above to obtain the estimate (2.4), though now with

$$a(s,8) = \frac{7}{8} - \frac{s-144}{384}$$

and

$$b(s,8) = \frac{7}{8} - \frac{s-168}{192}.$$

One may confirm that $b(s,8) \geqslant a(s,8)$ when $s \leqslant 192$, and otherwise $b(s,8) < a(s,8)$. Hence, by summing over dyadic intervals, we conclude that

$$\widetilde{E}_{s,8}(N) \ll N^{\omega_{s,8}+\varepsilon}\psi_8(N)^4,$$

when $164 < s \leqslant 180$, where $\omega_{s,8} = \frac{7}{8} - \frac{s-168}{192}$. Note that when $s \leqslant 170$, the latter bound is weaker than what follows by appropriate application of the methods of [7] and [1] (see the discussion in the introduction following the statement of Theorem 1.6). In this way, we have confirmed the upper bound asserted in Theorem 1.6 for $171 \leqslant s \leqslant 180$.

3. Sums of five cubes

Our goal in this section is the proof of Theorem 1.7, and to this end we adapt the treatment of §3 of Brüdern and Wooley [4] so as to incorporate the estimate supplied by Lemma 2.1 above. We begin by fixing some notation. We take N to be a large positive number, and write $P = N^{1/3}$. Also, define

$$f(\alpha) = \sum_{1 \leqslant x \leqslant P} e(\alpha x^3).$$

We define the minor arcs $\mathfrak{n}$ to be the set of points $\alpha \in [0,1)$ satisfying the property that whenever $q \in \mathbb{N}$, and $q\alpha$ differs from an integer by at most $P^{-9/4}$, then $q > P^{3/4}$. In addition, write $\mathfrak{N} = [0,1) \setminus \mathfrak{n}$. By orthogonality, when $n \leqslant N$, one has

$$R_{5,3}(n) = \int_0^1 f(\alpha)^5 e(-n\alpha)\,d\alpha.$$

We put

$$E(n) = R_{5,3}(n) - \frac{\Gamma(\frac{4}{3})^5}{\Gamma(\frac{5}{3})}\mathfrak{S}_{5,3}(n)n^{2/3},$$

and examine the value distribution of $E(n)$. Standard methods familiar to aficionados of the circle method confirm that, whenever $0 < \delta < \frac{1}{12}$ and $1 \leqslant n \leqslant N$, one has

$$\int_{\mathfrak{N}} f(\alpha)^5 e(-n\alpha)\,d\alpha = \frac{\Gamma(\frac{4}{3})^5}{\Gamma(\frac{5}{3})}\mathfrak{S}_{5,3}(n)n^{2/3} + O(N^{2/3-\delta}).$$

Such a conclusion may be found in Lemma 2.1 of [4], for example. In addition, as a consequence of Theorem 4.3 of [14], one has $\mathfrak{S}_{5,3}(n) \ll 1$. Thus, we may

infer that when h is positive and $0 < \delta < \frac{1}{12}$, one has

$$\sum_{N/2<n\leqslant N} |E(n)|^h \ll M_h(N) + O(N^{1+2h/3-h\delta}), \tag{3.1}$$

in which we have written

$$M_h(N) = \sum_{N/2<n\leqslant N} \Big| \int_{\mathfrak{n}} f(\alpha)^5 e(-n\alpha)\, d\alpha \Big|^h. \tag{3.2}$$

Theorem 3.1. *When $h \geqslant 2$, one has*

$$M_h(N) \ll N^\varepsilon (N^{\frac{11}{12}h} + N^{\frac{8}{9}h+\frac{2}{9}} + N^{\frac{3}{4}h+\frac{2}{3}}).$$

Proof. In order to bound the minor arc moments in (3.2), we follow the procedure described at the beginning of §3 of [4]. When T is a positive number, let $\mathcal{Z}_T(N)$ denote the set of natural numbers n with $1 \leqslant n \leqslant N$ for which one has the lower bound

$$\Big| \int_{\mathfrak{n}} f(\alpha)^5 e(-n\alpha)\, d\alpha \Big| > T.$$

Also, put $\mathcal{Z}_T^*(N) = \mathcal{Z}_T(N) \setminus \mathcal{Z}_{2T}(N)$. We aim to bound $Z_T = \operatorname{card}(\mathcal{Z}_T^*(N))$. Define the complex numbers η_n by putting $\eta_n = 0$ for $n \notin \mathcal{Z}_T^*(N)$, and when $n \in \mathcal{Z}_T^*(N)$ by means of the equation

$$\Big| \int_{\mathfrak{n}} f(\alpha)^5 e(-n\alpha)\, d\alpha \Big| = \eta_n \int_{\mathfrak{n}} f(\alpha)^5 e(-n\alpha)\, d\alpha.$$

We note that $|\eta_n| = 1$ when $\eta_n \neq 0$, and moreover

$$Z_T T \leqslant \sum_{1\leqslant n\leqslant N} \eta_n \int_{\mathfrak{n}} f(\alpha)^5 e(-n\alpha)\, d\alpha = \int_{\mathfrak{n}} f(\alpha)^5 K_T(-\alpha)\, d\alpha, \tag{3.3}$$

where

$$K_T(\alpha) = \sum_{1\leqslant n\leqslant N} \eta_n e(n\alpha).$$

We first estimate the integral on the right hand side of (3.3) by means of Lemma 2.1 above. On considering the underlying diophantine equations, one finds that

$$\int_0^1 |f(\alpha)^2 K_T(\alpha)^2|\, d\alpha \ll P^\varepsilon (P Z_T + P^{1/2} Z_T^{3/2}). \tag{3.4}$$

From Hua's lemma (see Lemma 2.5 of [14]), on the other hand, one has

$$\int_0^1 |f(\alpha)|^8\, d\alpha \ll P^{5+\varepsilon}.$$

An application of Hölder's inequality to (3.3) therefore reveals that

$$\begin{aligned} Z_T T &\leqslant \Big(\int_0^1 |f(\alpha)^2 K_T(\alpha)^2|\, d\alpha \Big)^{1/2} \Big(\int_0^1 |f(\alpha)|^8\, d\alpha \Big)^{1/2} \\ &\ll P^{5/2+\varepsilon} (P Z_T + P^{1/2} Z_T^{3/2})^{1/2}, \end{aligned}$$

whence

$$Z_T \ll P^\varepsilon (P^6 T^{-2} + P^{11} T^{-4}).$$

By dividing the range of summation into dyadic intervals, therefore, it follows that when $\nu > 0$ and $2 \leqslant h \leqslant 4$, one has

$$\sum_{\substack{n \in \mathcal{Z}_T^*(N) \\ P^{9/4} \leqslant T \leqslant P^{8/3+\nu}}} \left| \int_{\mathfrak{n}} f(\alpha)^5 e(-n\alpha)\, d\alpha \right|^h \ll P^\varepsilon \left(P^6 (P^{8/3+\nu})^{h-2} + P^{11} (P^{9/4})^{h-4} \right)$$
$$\ll P^\varepsilon (P^{\frac{8}{3}h + \frac{2}{3} + (h-2)\nu} + P^{\frac{9}{4}h+2}). \tag{3.5}$$

Meanwhile, when $h > 4$, one instead obtains

$$\sum_{\substack{n \in \mathcal{Z}_T^*(N) \\ P^{9/4} \leqslant T \leqslant P^{8/3+\nu}}} \left| \int_{\mathfrak{n}} f(\alpha)^5 e(-n\alpha)\, d\alpha \right|^h \ll P^\varepsilon \left(P^6 (P^{8/3+\nu})^{h-2} + P^{11} (P^{8/3+\nu})^{h-4} \right)$$
$$\ll P^{\frac{8}{3}h + \frac{2}{3} + (h-2)\nu + \varepsilon}. \tag{3.6}$$

Next we recall equation (3.8) of [4], which supplies the estimate

$$\sum_{n \in \mathcal{Z}_T^*(N)} \left| \int_{\mathfrak{n}} f(\alpha)^5 e(-n\alpha)\, d\alpha \right|^h \ll P^{13/2+\varepsilon} T^{h-2}.$$

Again dividing the range of summation into dyadic intervals, we find that

$$\sum_{\substack{n \in \mathcal{Z}_T^*(N) \\ 0 \leqslant T \leqslant P^{9/4}}} \left| \int_{\mathfrak{n}} f(\alpha)^5 e(-n\alpha)\, d\alpha \right|^h \ll P^{13/2+\varepsilon} (P^{9/4})^{h-2}$$
$$\ll P^{\frac{9}{4}h + 2 + \varepsilon}. \tag{3.7}$$

Finally, we recall equations (3.12) and (3.13) of [4], so that we have available the estimates

$$\int_{\mathfrak{n}} f(\alpha)^5 e(-n\alpha)\, d\alpha \ll P^{11/4} \tag{3.8}$$

and

$$\sum_{\substack{n \in \mathcal{Z}_T^*(N) \\ P^{8/3+\nu} \leqslant T \leqslant P^{11/4}}} \left| \int_{\mathfrak{n}} f(\alpha)^5 e(-n\alpha)\, d\alpha \right|^h \ll P^{11/2} (P^{11/4})^{h-2} \ll P^{\frac{11}{4}h}. \tag{3.9}$$

In order to confirm the conclusion of Theorem 3.1, we have only to recall that $P = N^{1/3}$, note (3.8), and collect together the estimates (3.5), (3.6), (3.7) and (3.9). In this way we conclude that

$$\sum_{n \in \mathcal{Z}_T^*(N)} \left| \int_{\mathfrak{n}} f(\alpha)^5 e(-n\alpha)\, d\alpha \right|^h \ll N^\varepsilon (N^{\frac{3}{4}h + \frac{2}{3}} + N^{\frac{8}{9}h + \frac{2}{9} + (h-2)\nu} + N^{\frac{11}{12}h}).$$

The desired conclusion then follows by taking ν sufficiently small, though positive. $\square$

Returning to the relation (3.1), we now take ν to be any positive number with $\nu < \frac{1}{5}$, and put $h = \frac{7}{2} - \nu$. Theorem 3.1 yields the estimate

$$\begin{aligned} M_h(N) &\ll N^{\frac{2}{3}h+1+\varepsilon}(N^{\frac{1}{4}h-1} + N^{\frac{2}{9}h-\frac{7}{9}} + N^{\frac{1}{12}h-\frac{1}{3}}) \\ &\ll N^{\frac{2}{3}h+1-\nu/6}. \end{aligned}$$

On recalling (3.1) and summing over dyadic intervals, we conclude that whenever h is a positive number smaller than $\frac{7}{2}$, and $0 < \delta < \frac{1}{12}(7-2h)$, then

$$\sum_{1\leqslant n\leqslant N} |E(n)|^h \ll N^{2h/3+1-\delta}.$$

This completes the proof of the first estimate of Theorem 1.7.

The second estimate of Theorem 1.7 follows from the case $h = 3$ of Theorem 3.1, which delivers the bound

$$M_3(N) \ll N^{35/12+\varepsilon}(N^{-1/6} + N^{-1/36} + 1).$$

The desired conclusion therefore follows from (3.1) by summing over dyadic intervals, since in that asymptotic relation one may take δ to be any positive number smaller than $\frac{1}{12}$.

4. A twelfth moment of cubic Weyl sums

We turn our attention in this section to the problem of establishing the estimate for the twelfth moment of cubic Weyl sums claimed in Theorem 1.8. Some preliminary manoeuvres are required to set the scene. Recall the notation and hypotheses of the statement of Theorem 1.8. These hypotheses ensure that λ_1, λ_2 and λ_3 are linearly independent, and thus there are non-zero integers A, B and C, depending at most on $\mathbf{c}$ and $\mathbf{d}$, with the property that $(A,B,C) = 1$ and $C\lambda_3 = A\lambda_1 + B\lambda_2$. Write

$$\mathcal{F}(\theta) = |F(\theta)^2 h(\theta)^2|,$$

and then put

$$\Theta_{\boldsymbol{\lambda}}(P) = \int_0^1 \int_0^1 \mathcal{F}(\lambda_1)\mathcal{F}(\lambda_2)\mathcal{F}(\lambda_3)\, d\alpha\, d\beta. \tag{4.1}$$

Then on making use of the periodicity of the integrand on the right hand side of (4.1), and changing variables, one finds that

$$\begin{aligned} \Theta_{\boldsymbol{\lambda}}(P) &= C^{-2}\int_0^C \int_0^C \mathcal{F}(\lambda_1)\mathcal{F}(\lambda_2)\mathcal{F}(\lambda_3)\, d\alpha\, d\beta \\ &= \int_0^1 \int_0^1 \mathcal{F}(C\lambda_1)\mathcal{F}(C\lambda_2)\mathcal{F}(A\lambda_1 + B\lambda_2)\, d\alpha\, d\beta \\ &\ll \int_0^1 \int_0^1 \mathcal{F}(C\theta)\mathcal{F}(C\phi)\mathcal{F}(A\theta + B\phi)\, d\theta\, d\phi. \end{aligned}$$

Next, we write $R(n)$ for the number of representations of an integer n in the shape

$$n = x_1^3 - x_2^3 + x_3^3 - x_4^3,$$

with $P/2 \leqslant x_1, x_2 \leqslant P$ and $x_3, x_4 \in \mathcal{A}(P,R)$. Then, on considering the underlying diophantine equations, one finds that

$$\Theta_{\boldsymbol{\lambda}}(P) \ll \sum_{|n_1|\leqslant 2P^3} \sum_{|n_2|\leqslant 2P^3} \sum_{|n_3|\leqslant 2P^3} R(n_1)R(n_2)R(n_3),$$

in which the summation is restricted by the conditions

$$Cn_1 = An_3 \quad \text{and} \quad Cn_2 = Bn_3. \tag{4.2}$$

For suitable non-zero integers a, b, c, one finds that the integers $\mathbf{n}$ solving the system (4.2) take the shape $\mathbf{n} = (ak, bk, ck)$, for some $k \in \mathbb{Z}$. We therefore find from Hölder's inequality that

$$\Theta_{\boldsymbol{\lambda}}(P) \ll \sum_{\substack{|dk|\leqslant 2P^3\\(d=a,b,c)}} R(ak)R(bk)R(ck) \leqslant (J_aJ_bJ_c)^{1/3},$$

where

$$J_d = \sum_{|dk|\leqslant 2P^3} R(dk)^3 \quad (d = a, b, c).$$

Define the exponential sum $g(\alpha) = g(\alpha; P)$ by

$$g(\alpha; P) = \sum_{P/2<x\leqslant P} e(\alpha x^3).$$

Then, on considering the underlying diophantine equations, one finds that

$$J_d \leqslant \sum_{|k|\leqslant 2P^3} R(k)^3 = \sum_{|k|\leqslant 2P^3} \Big(\int_0^1 |g(\theta)^2 h(\theta)^2| e(-\theta k)\, d\theta \Big)^3.$$

We therefore conclude at this point that

$$\Theta_{\boldsymbol{\lambda}}(P) \ll \sum_{|k|\leqslant 2P^3} \Big(\int_0^1 |g(\theta)^2 h(\theta)^2| e(-\theta k)\, d\theta \Big)^3. \tag{4.3}$$

Define the sets of arcs $\mathfrak{N}$ and $\mathfrak{n}$ as in section 3. Then from Lemma 3.4 of [2], one finds that

$$\int_{\mathfrak{N}} |g(\theta)^2 h(\theta)^2|\, d\theta \ll P^{1+\varepsilon}. \tag{4.4}$$

We therefore deduce from (4.3) that

$$\Theta_{\boldsymbol{\lambda}}(P) \ll \sum_{|k|\leqslant 2P^3} \Big| \int_{\mathfrak{n}} |g(\theta)^2 h(\theta)^2| e(-\theta k)\, d\theta \Big|^3 + O(P^{6+\varepsilon}). \tag{4.5}$$

We now let $\mathcal{Z}_T(P)$ denote the set of integers k with $|k| \leqslant 2P^3$ for which one has the lower bound

$$\Big| \int_{\mathfrak{n}} |g(\theta)^2 h(\theta)^2| e(-\theta k)\, d\theta \Big| > T.$$

Also, we put $\mathcal{Z}_T^*(P) = \mathcal{Z}_T(P) \setminus \mathcal{Z}_{2T}(P)$ and $Z_T = \mathrm{card}(\mathcal{Z}_T^*(P))$. Define the complex numbers η_k by putting $\eta_k = 0$ for $k \notin \mathcal{Z}_T^*(P)$, and when $k \in \mathcal{Z}_T^*(P)$ by means of the equation

$$\Bigl|\int_{\mathfrak{n}} |g(\theta)^2 h(\theta)^2| e(-\theta k)\,d\theta\Bigr| = \eta_k \int_{\mathfrak{n}} |g(\theta)^2 h(\theta)^2| e(-\theta k)\,d\theta.$$

Again, we have $|\eta_k| = 1$ whenever $\eta_k \neq 0$, and moreover

$$\begin{aligned} Z_T T &\leqslant \sum_{|k|\leqslant 2P^3} \eta_k \int_{\mathfrak{n}} |g(\theta)^2 h(\theta)^2| e(-\theta k)\,d\theta \\ &= \int_{\mathfrak{n}} |g(\theta)^2 h(\theta)^2| K_T(-\theta)\,d\theta, \end{aligned} \tag{4.6}$$

where

$$K_T(\theta) = \sum_{|k|\leqslant 2P^3} \eta_k e(k\theta).$$

We estimate the integral on the right hand side of (4.6) through the medium of Lemma 2.1. The estimate (3.4) again holds in the present context as a consequence of the latter lemma. Also, on considering the underlying diophantine equations, from Theorem 1.2 of [16] one has

$$\int_0^1 |g(\theta)^2 h(\theta)^4|\,d\theta \ll P^{3+\xi+\varepsilon},$$

where $\xi = \frac{1}{4} - \tau$. By applying Hölder's inequality to (4.6) and considering the underlying diophantine equations, we therefore deduce that

$$\begin{aligned} Z_T T &\leqslant \Bigl(\int_0^1 |g(\theta)^2 K_T(\theta)^2|\,d\theta\Bigr)^{1/2} \Bigl(\int_0^1 |g(\theta)^2 h(\theta)^4|\,d\theta\Bigr)^{1/2} \\ &\ll P^\varepsilon (P Z_T + P^{1/2} Z_T^{3/2})^{1/2} (P^{3+\xi})^{1/2}, \end{aligned}$$

whence

$$Z_T \ll P^{4+\xi+\varepsilon} T^{-2} + P^{7+2\xi+\varepsilon} T^{-4}.$$

Let ν be a small positive number. Then by dividing the range of summation into dyadic intervals, we obtain the estimate

$$\begin{aligned} &\sum_{\substack{k\in\mathcal{Z}_T^*(P) \\ P^{5/4+\xi/2}\leqslant T\leqslant P^{11/6+\nu}}} \Bigl|\int_{\mathfrak{n}} |g(\theta)^2 h(\theta)^2| e(-\theta k)\,d\theta\Bigr|^3 \\ &\qquad \ll P^\varepsilon \Bigl(P^{4+\xi}(P^{\frac{11}{6}+\nu}) + P^{7+2\xi}(P^{\frac{5}{4}+\frac{\xi}{2}})^{-1}\Bigr) \\ &\qquad \ll P^{\frac{23}{4}+\frac{3\xi}{2}+\varepsilon}. \end{aligned} \tag{4.7}$$

Next, we recall that a modified version of Weyl's inequality yields the bound

$$\sup_{\theta\in\mathfrak{n}} |g(\theta)| \ll P^{3/4+\varepsilon}$$

(see, for example, Lemma 1 of [12]). Then by Schwarz's inequality and Parseval's identity, one obtains from (4.6) the upper bound

$$Z_T T \leqslant \Big(\sup_{\theta\in\mathfrak{n}} |g(\theta)|\Big)\Big(\int_0^1 |g(\theta)^2h(\theta)^4|\, d\theta\Big)^{1/2}\Big(\int_0^1 |K_T(\theta)|^2\, d\theta\Big)^{1/2}$$
$$\ll P^{3/4+\varepsilon}(P^{3+\xi})^{1/2}Z_T^{1/2},$$

whence

$$Z_T \ll P^{9/2+\xi+\varepsilon}T^{-2}.$$

Dividing the range of summation once again into dyadic intervals, we see now that

$$\sum_{\substack{k\in\mathcal{Z}_T^*(P)\\ 0\leqslant T\leqslant P^{5/4+\xi/2}}} \Big|\int_{\mathfrak{n}} |g(\theta)^2h(\theta)^2|e(-\theta k)\, d\theta\Big|^3 \ll P^{\frac{9}{2}+\xi+\varepsilon}(P^{\frac{5}{4}+\frac{\xi}{2}})$$
$$\ll P^{\frac{23}{4}+\frac{3\xi}{2}+\varepsilon}. \tag{4.8}$$

Finally, as a consequence of Hooley's work [8] on sums of four cubes, one has

$$\int_0^1 |g(\theta)^2h(\theta)^2|e(-\theta k)\, d\theta \ll P^{11/6+\varepsilon}$$

whenever $k \neq 0$ (see Lemma 2.1 of Parsell [11]). On recalling (4.4), it follows that when k is non-zero, one has

$$\int_{\mathfrak{n}} |g(\theta)^2h(\theta)^2|e(-\theta k)\, d\theta \ll P^{11/6+\varepsilon} + P^{1+\varepsilon} \ll P^{11/6+\varepsilon}.$$

When $k = 0$, meanwhile, it follows from Hua's lemma (see Lemma 2.5 of [14]) that

$$\int_0^1 |g(\theta)^2h(\theta)^2|\, d\theta \ll P^{2+\varepsilon}.$$

We therefore deduce that

$$\sum_{\substack{k\in\mathcal{Z}_T^*(P)\\ T>P^{11/6+\nu}}} \Big|\int_{\mathfrak{n}} |g(\theta)^2h(\theta)^2|e(-\theta k)\, d\theta\Big|^3 = \Big(\int_{\mathfrak{n}} |g(\theta)^2h(\theta)^2| d\theta\Big)^3$$
$$\ll (P^{2+\varepsilon})^3. \tag{4.9}$$

Combining (4.7), (4.8) and (4.9), we find that

$$\sum_{k\in\mathcal{Z}_T^*(P)} \Big|\int_{\mathfrak{n}} |g(\theta)^2h(\theta)^2|e(-\theta k)\, d\theta\Big|^3 \ll P^{\frac{23}{4}+\frac{3\xi}{2}+\varepsilon},$$

so that on summing over dyadic intervals, we deduce from (4.5) that

$$\Theta_{\boldsymbol{\lambda}}(P) \ll P^{\frac{23}{4}+\frac{3\xi}{2}+\varepsilon}. \tag{4.10}$$

The first estimate of Theorem 1.8 now follows on recalling the definition of $\Theta_{\boldsymbol{\lambda}}(P)$.

In order to confirm the second estimate of Theorem 1.8, we begin by making a dyadic dissection of the smooth Weyl sum $h(\alpha)$. When $1 \leqslant R \leqslant Q$, write

$\mathcal{B}(Q,R)=\mathcal{A}(Q,R)\setminus\mathcal{A}(Q/2,R)$, and define the exponential sum $H(\alpha;Q)=H(\alpha;Q,R)$ by

$$H(\alpha;Q,R)=\sum_{x\in\mathcal{B}(Q,R)}e(\alpha x^3).$$

We suppose throughout that $1\leqslant R\leqslant P^\eta$. Then on putting $L=[\frac12\log P]$, we find that

$$|h(\alpha;P,R)|\leqslant\sum_{l=0}^{L}|H(\alpha;2^{-l}P)|+O(\sqrt{P}).$$

As a consequence of Hua's lemma (see Lemma 2.5 of [14]), when $0\leqslant l\leqslant L$ and $1\leqslant i\leqslant 3$, one has

$$\int_0^1\int_0^1|H(\lambda_i;2^{-l}P)|^4\,d\alpha\,d\beta\leqslant\int_0^1|H(\theta;2^{-l}P)|^4\,d\theta\ll P^{2+\varepsilon},$$

and when $0\leqslant l,m\leqslant L$ and $1\leqslant i<j\leqslant 3$, one has

$$\int_0^1\int_0^1|H(\lambda_i;2^{-l}P)H(\lambda_j;2^{-m}P)|^4\,d\alpha\,d\beta$$
$$\leqslant\int_0^1\int_0^1|H(\theta;2^{-l}P)H(\phi;2^{-m}P)|^4\,d\theta\,d\phi\ll P^{4+\varepsilon}.$$

Thus we deduce that

$$\int_0^1\int_0^1|h(\lambda_1)h(\lambda_2)h(\lambda_3)|^4\,d\alpha\,d\beta\ll P^\varepsilon\Theta'_{\boldsymbol\lambda}(P)+O(P^{6+\varepsilon}),\tag{4.11}$$

where

$$\Theta'_{\boldsymbol\lambda}(P)=\max_{\sqrt{P}\leqslant Q_1,Q_2,Q_3\leqslant P}\int_0^1\int_0^1|H(\lambda_1;Q_1)H(\lambda_2;Q_2)H(\lambda_3;Q_3)|^4\,d\alpha\,d\beta.$$

From here, the argument applied above leading from (4.1) to (4.3) may be applied, mutatis mutandis, and thereby we establish via Hölder's inequality that

$$\Theta'_{\boldsymbol\lambda}(P)\ll\max_{\sqrt{P}\leqslant Q\leqslant P}\sum_{|k|\leqslant 2Q^3}\Bigl(\int_0^1|H(\theta;Q)|^4e(-\theta k)\,d\theta\Bigr)^3.$$

Hence, on considering the underlying diophantine equations, one finds that

$$\Theta'_{\boldsymbol\lambda}(P)\ll\max_{\sqrt{P}\leqslant Q\leqslant P}\sum_{|k|\leqslant 2Q^3}\Bigl(\int_0^1|g(\theta;Q)^2h(\theta;Q,Q^{3\eta})^2|e(-\theta k)\,d\theta\Bigr)^3.$$

A comparison between the sum on the right hand side of this estimate, with that on the right hand side of (4.3), reveals that the argument employed above to deliver (4.10) in this instance shows that

$$\Theta'_{\boldsymbol\lambda}(P)\ll\max_{\sqrt{P}\leqslant Q\leqslant P}Q^{\frac{23}{4}+\frac{3\xi}{2}+\varepsilon}\leqslant P^{\frac{23}{4}+\frac{3\xi}{2}+\varepsilon}.$$

The second estimate of Theorem 1.8 consequently follows from (4.11).

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KK: Department of Mathematics, Faculty of Education, Iwate University, Morioka, 020-8550 Japan

E-mail address: `kawada@iwate-u.ac.jp`

TDW: School of Mathematics, University of Bristol, University Walk, Clifton, Bristol BS8 1TW, United Kingdom

E-mail address: `matdw@bristol.ac.uk`